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FAST CONVERGING SERIES FOR SOME FAMOUS CONSTANTS

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Abstract. For $k = 0, 1, 2, \ldots$ let $H_k^{(2)} := \sum_{0<j\leqslant k} 1/j^2$ and $\bar H_k^{(2)} := \sum_{0<j\leqslant k} 1/(2j-1)^2$. In this paper we give a new type of exponentially converging series (involving $H_k^{(2)}$ or $\bar H_k^{(2)}$) for certain famous constants. For example, we show that

$$\sum_{k=1}^\infty \frac{H_{k-1}^{(2)}}{k^2\binom{2k}{k}} = \frac{\pi^4}{1944},\quad \sum_{k=1}^\infty \frac{\binom{2k}{k}\bar H_k^{(2)}}{(2k+1)(-16)^k} = -\frac{1}{3}\log^3\phi,$$

$$\sum_{k=1}^\infty \frac{L_{2k}H_{k-1}^{(2)}}{k^2\binom{2k}{k}} = \frac{41\pi^4}{7500},\quad \sum_{k=1}^\infty \frac{\binom{2k}{k}L_{2k+1}\bar H_k^{(2)}}{(2k+1)16^k} = \frac{13\pi^3}{1500},$$

where ϕ is the golden ratio $(\sqrt5+1)/2$, and $L_0, L_1, L_2, \ldots$ are Lucas numbers given by $L_0 = 2,\ L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$ $(n = 1, 2, 3, \ldots)$.

1. Introduction

Series for powers of π or $\log 2$ have a long history (see, e.g., [M]). It is well known that

$$\sum_{k=0}^\infty \frac{(-1)^k}{2k+1} = \frac{\pi}{4},\quad \zeta(2) := \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6},\quad \zeta(4) := \sum_{k=1}^\infty \frac{1}{k^4} = \frac{\pi^4}{90}$$

and

$$\log 2 = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k}.$$

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However, these series converge very slow. In 1974 Gosper announced the following fast converging series for π (cf. G. Almkvist, C. Krattenthaler and J. Petersson [AKP]):

$$\sum_{k=0}^{\infty}\frac{25k-3}{2^k\binom{3k}{k}}=\frac{\pi}{2}.$$

(Note that $n!\sim\sqrt{2\pi n}(n/e)^n\quad(n\to+\infty)$ by Stirling's formula.) In 1993 D. Zeilberger [Z] obtained the identity

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{6}$$

via the WZ method (see, e.g., [PWZ]). This is a rapidly converging series for π^2 since

$$\binom{2k}{k}\sim\frac{4^k}{\sqrt{k\pi}}\quad(k\to+\infty)$$

by Stirling's formula. In 2006 D. Bailey, J. Borwein and D. Bradley [BBB] obtained the celebrated formula

$$\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}=\frac{17}{36}\zeta(4)=\frac{17\pi^4}{3240}.$$

Here is a fast converging series for $\log 2$ given by Borwein and Bailey [BB. p. 129]:

$$\log 2=\frac{2}{3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)9^k}.$$

Recall that harmonic numbers are those rational numbers

$$H_n:=\sum_{0<k\leqslant n}\frac{1}{k}\qquad(n=0,1,2,\ldots).$$

and *harmonic numbers of the second order* are defined by

$$H_n^{(2)}:=\sum_{0<k\leqslant n}\frac{1}{k^2}\qquad(n=0,1,2,\ldots).$$

Recently the author [S6] found and proved that

$$\sum_{k=1}^{\infty}\frac{2^kH_{k-1}^{(2)}}{k\binom{2k}{k}}=\frac{\pi^3}{48},$$

which was motivated by his philosophy of correspondence between certain series and congruences (cf. [S5]) and his related result

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{2^k} H_k^{(2)} \equiv -E_{p-3} \pmod p \quad \text{for any prime } p > 3,$$

where $E_0, E_1, E_2, \ldots$ are Euler numbers given by $E_0 = 1$ and the recursion

$$\sum_{\substack{k=0 \\ 2\mid k}}^{n} \binom{n}{k} E_{n-k} = 0 \qquad (n = 1, 2, 3, \ldots).$$

The reader may consult [PS], [S1-6] and [ST1, ST2] for congruences involving central binomial coefficients which provide important backgrounds for our current research on series. In [S6] the author explained how he was led to consider $\sum_{k=0}^{p-1} \binom{2k}{k} H_k^{(2)}/2^k$ modulo an odd prime p and the evaluation of the series $\sum_{k=1}^{\infty} 2^k H_{k-1}^{(2)}/(k\binom{2k}{k})$. Note that $H_n^{(2)} \to \zeta(2) = \pi^2/6$ as $n \to +\infty$.

In this paper we aim at giving certain exponentially converging series for some classical constants (such as π^3 and π^4) with each term related to the second-order harmonic numbers.

Recall that the Fibonacci sequence $\{F_n\}_{n\geqslant 0}$ and the Lucas sequence $\{L_n\}_{n\geqslant 0}$ are given by

$$F_0 = 0,\ F_1 = 1, \text{ and } F_{n+1} = F_n + F_{n-1}\ (n = 1, 2, 3, \ldots),$$

and

$$L_0 = 2,\ L_1 = 1, \text{ and } L_{n+1} = L_n + L_{n-1}\ (n = 1, 2, 3, \ldots).$$

It is well known that for each $n = 0, 1, 2, \ldots$ we have

$$\sqrt{5}F_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \text{ and } L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Now we state our first theorem in which we relate Fibonacci numbers and Lucas numbers to π^4 for the first time.

Theorem 1.1. (i) *We have the new identity*

$$\sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)}}{k^2\binom{2k}{k}} = \frac{\pi^4}{1944}. \tag{1.1}$$

Also,

$$\sum_{k=1}^\infty \frac{L_{2k}H_{k-1}^{(2)}}{k^2\binom{2k}{k}}=\frac{41\pi^4}{7500},\quad \sum_{k=1}^\infty \frac{F_{2k}H_{k-1}^{(2)}}{k^2\binom{2k}{k}}=\frac{2\pi^4}{375\sqrt5},\tag{1.2}$$

and

$$\sum_{k=1}^\infty \frac{v_kH_{k-1}^{(2)}}{k^2\binom{2k}{k}}=\frac{34\pi^4}{1875}\quad \text{and}\quad \sum_{k=1}^\infty \frac{u_kH_{k-1}^{(2)}}{k^2\binom{2k}{k}}=\frac{2\pi^4}{125\sqrt5},\tag{1.3}$$

where

$$u_k=\begin{cases}5^{k/2}F_k & \text{if } 2\mid k,\\ 5^{(k-1)/2}L_k & \text{if } 2\nmid k,\end{cases}\quad \text{and}\quad v_k=\begin{cases}5^{k/2}L_k & \text{if } 2\mid k,\\ 5^{(k+1)/2}F_k & \text{if } 2\nmid k.\end{cases}$$

(ii) *We also have*

$$\sum_{k=1}^\infty \frac{H_{k-1}^{(2)}x^k}{k\binom{2k}{k}}=\frac43\sqrt{\frac{x}{4-x}}\arcsin^3\frac{\sqrt x}{2}\quad \text{for } 0\leqslant x<4.\tag{1.4}$$

In particular,

$$\sum_{k=1}^\infty \frac{H_{k-1}^{(2)}}{k\binom{2k}{k}}=\frac{\pi^3}{162\sqrt3}\tag{1.5}$$

and

$$\sum_{k=1}^\infty \frac{3^kH_{k-1}^{(2)}}{k\binom{2k}{k}}=\frac{4\pi^3}{27\sqrt3}.\tag{1.6}$$

Remark 1.1. (a) The modern `Mathematica 7` (version 7) employs various powerful algorithms (including the WZ method) to obtain summations of series, however it is still unable to evaluate the series in Theorem 1.1. (b) Motivated by massive Feynman diagrams in physics, A. I. Davydychev and M. Yu. Kalmykov [DK, (C16) and (C17)] obtained that for any $x>1/4$ we have

$$\sum_{k=1}^\infty \frac{H_{k-1}^{(2)}}{k\binom{2k}{k}(-x)^k}=\frac{1}{6\sqrt{4x+1}}\log^3\frac{\sqrt{4x+1}+1}{\sqrt{4x+1}-1}$$

and

$$\sum_{k=1}^\infty \frac{H_{k-1}^{(2)}}{k^2\binom{2k}{k}(-x)^k}=\frac{1}{24}\log^4\frac{\sqrt{4x+1}+1}{\sqrt{4x+1}-1}.$$

For convenience, we set

$$\bar H_n^{(2)}:=\sum_{0<k\leqslant n}\frac{1}{(2k-1)^2}=H_{2n}^{(2)}-\frac{H_n^{(2)}}{4}\quad \text{for } n=0,1,2,\ldots.$$

Our second theorem and its corollaries deal with exponentially converging series for π^3, π^2 or $\log^3 x$ with the kth term related to $\bar H_k^{(2)}$.

Theorem 1.2. (i) *If* $0 < x \leqslant 4$, *then*

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{2k+1}\left(\frac{x}{16}\right)^k=\frac{\arcsin^3(\sqrt{x}/2)}{3\sqrt{x}}. \tag{1.7}$$

Also,

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}F_{2k+1}\bar{H}_k^{(2)}}{(2k+1)16^k}=\frac{7\pi^3}{750\sqrt{5}} \tag{1.8}$$

and

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}L_{2k+1}\bar{H}_k^{(2)}}{(2k+1)16^k}=\frac{13\pi^3}{1500}. \tag{1.9}$$

(ii) *For* $x \geqslant 1$ *we have*

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{(2k+1)(-4x)^k}=\frac{\sqrt{x}}{6}\log^3\frac{\sqrt{x+1}-1}{\sqrt{x}}. \tag{1.10}$$

In view of (1.7) and (1.10), one can easily deduce the following consequence.

Corollary 1.1. *We have*

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{(2k+1)16^k}=\frac{\pi^3}{648}, \tag{1.11}$$

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{(2k+1)8^k}=\frac{\pi^3}{192\sqrt{2}}, \tag{1.12}$$

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{2k+1}\left(\frac{3}{16}\right)^k=\frac{\pi^3}{81\sqrt{3}}, \tag{1.13}$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{(2k+1)(-32)^k}=-\frac{\sqrt{2}}{24}\log^3 2, \tag{1.14}$$

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{(2k+1)(-12)^k}=-\frac{\sqrt{3}}{48}\log^3 3, \tag{1.15}$$

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{(2k+1)(-16)^k}=-\frac{1}{3}\log^3\frac{\sqrt{5}+1}{2}. \tag{1.16}$$

Remark 1.2. (1.11) also follows from Section 2 of [PP]. Our (1.16) provides a fast converging series for $\log^3\phi$ (at a geometric rate with ratio $-1/4$), where $\phi=(\sqrt{5}+1)/2$ is the famous golden ratio. It seems that there are no previously known exponentially converging series for $\log^3\phi$.

Corollary 1.2. *For $0 \leqslant x < 2$ we have*

$$\sum_{k=1}^{\infty}\binom{2k}{k}\bar{H}_k^{(2)}\left(\frac{x}{4}\right)^{2k} = \frac{\arcsin^2(x/2)}{\sqrt{4-x^2}}. \tag{1.17}$$

In particular,

$$\sum_{k=1}^{\infty}\binom{2k}{k}\frac{\bar{H}_k^{(2)}}{16^k} = \frac{\pi^2}{36\sqrt{3}}, \tag{1.18}$$

$$\sum_{k=1}^{\infty}\binom{2k}{k}\frac{\bar{H}_k^{(2)}}{8^k} = \frac{\pi^2}{16\sqrt{2}}, \tag{1.19}$$

$$\sum_{k=1}^{\infty}\binom{2k}{k}\frac{3^k\bar{H}_k^{(2)}}{16^k} = \frac{\pi^2}{9}. \tag{1.20}$$

To conclude this section, we raise a conjecture on related congruences.

Conjecture 1.1. *Let p be an odd prime.*

(i) *If $p > 3$ then*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}H_k^{(2)} \equiv \frac{2H_{p-1}}{3p^2} + \frac{76}{135}p^2B_{p-5} \pmod{p^3},$$

where $B_0, B_1, B_2, \ldots$ are well known Bernoulli numbers. Also,

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{8^k}\bar{H}_k^{(2)} \equiv \left(\frac{-2}{p}\right)\frac{E_{p-3}}{4} \pmod{p},$$

where $(-)$ denotes the Legendre symbol.

(ii) *If $p > 3$, then*

$$\sum_{k=1}^{p-1}(-2)^k\binom{2k}{k}H_k^{(2)} \equiv \frac{2}{3}q_p(2)^2 \pmod{p},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$. When $p > 5$, we have

$$\sum_{k=1}^{p-1}(-1)^k\binom{2k}{k}H_k^{(2)} \equiv \frac{5}{2}\left(\frac{p}{5}\right)\frac{F_{p-(\frac{p}{5})}^2}{p^2} \pmod{p}.$$

2. Proofs of Theorems 1.1-1.2 and Corollary 1.2

Proof of Theorem 1.1. By `Mathematica 7`, for $0 \leqslant x \leqslant 4$ we have

$$\sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)} x^k}{k^2 \binom{2k}{k}} = \frac{2}{3} \arcsin^4 \frac{\sqrt{x}}{2}. \tag{2.1}$$

Putting $x = 1$ in (2.1) we get (1.1), however the series cannot be evaluated by `Mathematica 7` via the `FullSimplify` command.

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then

$$\arcsin \frac{\sqrt{\alpha^2}}{2} = \arcsin \frac{\sqrt{5}+1}{4} = \frac{3\pi}{10}$$

and

$$\arcsin \frac{\sqrt{\beta^2}}{2} = \arcsin \frac{\sqrt{5}-1}{4} = \frac{\pi}{10}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)} \alpha^{2k}}{k^2 \binom{2k}{k}} = \frac{2}{3} \left(\frac{3\pi}{10}\right)^4 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)} \beta^{2k}}{k^2 \binom{2k}{k}} = \frac{2}{3} \left(\frac{\pi}{10}\right)^4.$$

As $\alpha^{2k} + \beta^{2k} = L_{2k}$ and $\alpha^{2k} - \beta^{2k} = \sqrt{5} F_{2k}$, (1.2) follows.

It is easy to see that for any $k = 0, 1, 2, \ldots$ we have

$$\sqrt{5} u_k = \left(\frac{5+\sqrt{5}}{2}\right)^k - \left(\frac{5-\sqrt{5}}{2}\right)^k \text{ and } v_k = \left(\frac{5+\sqrt{5}}{2}\right)^k + \left(\frac{5-\sqrt{5}}{2}\right)^k.$$

Also,

$$\sin \frac{2\pi}{5} = \sqrt{1 - \left(\frac{\sqrt{5}-1}{4}\right)^2} = \frac{1}{2} \sqrt{\frac{5+\sqrt{5}}{2}}$$

and

$$\sin \frac{\pi}{5} = \sqrt{1 - \left(\frac{\sqrt{5}+1}{4}\right)^2} = \frac{1}{2} \sqrt{\frac{5-\sqrt{5}}{2}}.$$

Thus, applying (2.1) with $x = (5 \pm \sqrt{5})/2$ we obtain (1.3) immediately.

Since $\binom{2k}{k} \sim 4^k/\sqrt{k\pi}$ as $k \to \infty$, the series $\sum_{k=1}^{\infty} 4^k/(k^2\binom{2k}{k})$ converges. Note also that $H_{k-1}^{(2)} < \zeta(2) = \pi^2/6$ for $k = 1, 2, 3, \ldots$. Thus, we can take derivatives of both sides of (2.1) to get

$$\sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)} x^{k-1}}{k \binom{2k}{k}} = \frac{8}{3} \arcsin^3 \frac{\sqrt{x}}{2} \times \frac{1/(4\sqrt{x})}{\sqrt{1-x/4}} \quad \text{with } 0 \leqslant x < 4.$$

So (1.4) holds. Applying (1.4) with $x = 1, 3$ we obtain (1.5) and (1.6) respectively.

The proof of Theorem 1.1 is now complete. $\square$

Proof of Theorem 1.2. Let us recall a known hypergeometric identity

$$\sin(az) = a\sin(z)\cdot {}_2F_1\left(\frac{1+a}{2},\frac{1-a}{2},\frac{3}{2};\ \sin^2 z\right).$$

(See, e.g., [BE, §2.8].) For $k = 1, 2, 3, \ldots$, clearly

$$\begin{aligned}&\frac{\prod_{j=0}^{k-1}((1+a)/2+j)((1-a)/2+j)}{k!\prod_{j=1}^{k}(1/2+j)}\\ =&\frac{\prod_{j=0}^{k-1}((2j+1)^2-a^2)}{k!(2k+1)!!2^k} = \frac{\binom{2k}{k}}{(2k+1)4^k}\prod_{j=0}^{k-1}\left(1-\frac{a^2}{(2j+1)^2}\right).\end{aligned}$$

Therefore

$$\sin(az) = a\sin(z)\left(1+\sum_{k=1}^{\infty}\frac{\binom{2k}{k}}{2k+1}\left(\frac{\sin^2 z}{4}\right)^k\prod_{j=1}^{k}\left(1-\frac{a^2}{(2j-1)^2}\right)\right).$$

Comparing the coefficients of a^3 on both sides we get

$$\frac{z^3}{6} = \sin(z)\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{2k+1}\left(\frac{\sin^2 z}{4}\right)^k. \tag{2.2}$$

Now let $0 < x \leqslant 4$ and set $z = \arcsin(\sqrt{x}/2)$. Then (2.2) gives (1.7).

Applying (1.7) with $x = ((\sqrt{5}\pm 1)/2)^2$ we obtain

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{(2k+1)16^k}\left(\frac{\sqrt{5}\pm 1}{2}\right)^{2k+1} = \frac{1}{3}\left(\frac{(2\pm 1)\pi}{10}\right)^3$$

and hence (1.8) and (1.9) follow.

Now prove (1.10) for $x \geqslant 1$. Note that

$$\sin(iz) = \frac{e^{-z}-e^{z}}{2i} = i\sinh(z),$$

where $\sinh(z)$ is the hyperbolic sine function. Substituting iz for z in (2.2) we get

$$-\frac{z^3}{6} = \sinh(z)\sum_{k=1}^{\infty}\frac{\binom{2k}{k}\bar{H}_k^{(2)}}{2k+1}\left(-\frac{\sinh^2(z)}{4}\right)^k. \tag{2.3}$$

Fix $x \geqslant 1$ and let

$$\begin{aligned} z =& \operatorname{arcsinh}\frac{1}{\sqrt{x}} = \log\left(\frac{1}{\sqrt{x}} + \sqrt{1+\frac{1}{x}}\right) \\ =& \log\frac{\sqrt{x+1}+1}{\sqrt{x}} = \log\frac{\sqrt{x}}{\sqrt{x+1}-1} = -\log\frac{\sqrt{x+1}-1}{\sqrt{x}}. \end{aligned}$$

Then (2.3) yields (1.10).

So far we have completed the proof of Theorem 1.2. $\square$

Proof of Corollary 1.2. By Theorem 1.2(i), for $0 \leqslant x \leqslant 2$ we have

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}\bar{H}_k^{(2)} x^{2k+1}}{(2k+1)16^k} = \frac{1}{3}\arcsin^3\frac{x}{2}. \tag{2.4}$$

Observe that $\binom{2k}{k} < (1+1)^{2k} = 4^k$ and

$$\bar{H}_k^{(2)} < \sum_{j=1}^{\infty}\frac{1}{(2j-1)^2} = \sum_{j=1}^{\infty}\frac{1}{j^2} - \sum_{j=1}^{\infty}\frac{1}{(2j)^2} = \frac{\pi^2}{8}$$

for every $k = 1, 2, 3, \ldots$. Let $0 < \varepsilon \leqslant 2$. Clearly the series

$$S(\varepsilon) := \sum_{k=1}^{\infty}\frac{4^k(2-\varepsilon)^{2k+1}}{(2k+1)16^k}$$

converges, and

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}\bar{H}_k^{(2)} x^{2k+1}}{(2k+1)16^k} < \frac{\pi^2}{8}S(\varepsilon) \qquad \text{for } 0 \leqslant x \leqslant 2-\varepsilon.$$

Thus, we can take the derivatives of both sides of (2.4) with $0 \leqslant x \leqslant 2-\varepsilon$ to obtain (1.17). Therefore (1.17) holds for any $0 \leqslant x < 2$.

Applying (1.17) with $x = 1, \sqrt{2}, \sqrt{3}$ we get (1.18)-(1.20) immediately. This concludes the proof. $\square$

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